Power Series

Def: A series of the form

$$\sum_{n=0}^{\infty} a_n (x-\alpha)^n = a_0 + a_1(x-\alpha) + a_2(x-\alpha)^2 + \cdots$$

where $a_0, a_1, a_2, \ldots, a_n, \ldots$ are real constants and x is real variable is called power series about centre $x=\alpha$.

In particular: A series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \quad , \text{ is called}$$

power series in x about centre origin.

Remarks (i) Since on shifting origin to α, series $\sum_{n=0}^{\infty} a_n (x-\alpha)^n$ reduces to $\sum_{n=0}^{\infty} a_n x^n$, therefore we shall confine our studies to $\sum_{n=0}^{\infty} a_n x^n$.

(ii) The speciality of power series is that it either converges for all $x \in R$, or only for $x=0$ or in an interval from $-r$ to r, where r and $-r$ may or may not be included.

(iii) Power series about centre $x=\alpha$ is $\sum_{n=0}^{\infty} a_n (x-\alpha)^n$ always cgt about $x=\alpha$.

$\because$ at $x=\alpha$, $\sum_{n=0}^{\infty} a_0 (x-\alpha)^n = a_0$ (constant)

hence cgt. Every power series converges at least at one point i.e. about it centre.

Radius and Interval of convergence of power series:

For each power series $\sum_{n=0}^{\infty} a_n (x-\alpha)^n$, $\exists$ non-negative real no R s.t $\sum_{n=0}^{\infty} a_n (x-\alpha)^n$ is cgt for $|x-\alpha| < R$

i.e cgt for $x \in (\alpha-R, \alpha+R)$ and dgt for $|x-\alpha| > R$ i.e. dgt for $x \in (-\infty, \alpha-R, \alpha+R, \infty)$.

Then real no R is called radius of convergence for power series $\sum_{n=0}^{\infty} a_n (x-\alpha)^n$ and interval $(\alpha-R, \alpha+R)$ is called interval of convergence for power series $\sum_{n=0}^{\infty} a_n (x-\alpha)^n$.

Remarks: (i) for power series about origin i.e. $\sum_{n=0}^{\infty} a_n x^n$, $\exists$ non-negative real no. R s.t. series $\sum_{n=0}^{\infty} a_n x^n$ is cgt for $|x| < R$ i.e. $x \in (-R, R)$ and dgt out side this interval. Then interval $(-R, R)$ is called interval of convergence and real no R is called radius of convergence for $\sum a_n x^n$.

(ii) If R be radius of convergence for power series $\sum a_n x^n$, then $\sum a_n x^n$ is cgt for $|x| < R$ i.e. $x \in (-R, R)$ and dgt out side this interval. At $x = R$ or $-R$, series $\sum a_n x^n$ may or may not cgt. We have to check at these end points of interval for convergence separately.

Formulas to find Radius of convergence of power series:

If R be radius of convergence for power series $\sum a_n x^n$.

Then (i) $R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$ or $\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

(ii) $\frac{1}{R} = \lim_{n \to \infty} |a_n|^{1/n}$

or $R = \dfrac{1}{\lim_{n \to \infty} |a_n|^{1/n}}$

Th^m: For power series $\sum a_n x^n$, if $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = r$, r may be finite or infinite. Then $\frac{1}{r} = R$ is radius of convergence of $\sum a_n x^n$.

Sol^n: We have three cases arise.

Case (i) If $\lim_{n\to\infty} |a_n|^{1/n} = r = \infty$

$\therefore \lim_{n\to\infty} |a_n x^n|^{1/n} = 0$ iff $x=0$.

$\therefore \sum a_n x^n$ converges at $x=0$ only.

Hence $R = \frac{1}{r} = 0$ i.e. radius of convergence is zero.

Case (ii): If $\lim_{n\to\infty} |a_n|^{1/n} = r = 0$

$\Rightarrow \lim_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = 0 < 1 \quad \forall x \in R$.

$\therefore \sum a_n x^n$ converges for all $x \in R$.

$\therefore R = \frac{1}{r} = \infty$ i.e. radius of convergence is ∞.

Case (iii) Let $\lim_{n\to\infty} |a_n|^{1/n} = r$ (nonzero finite)

Here $\lim_{n\to\infty} |a_n x^n|^{1/n} = |x| r < 1$ iff $|x| < \frac{1}{r}$

$\therefore \sum a_n x^n$ converges for $|x| < \frac{1}{r} = R$ i.e. on $(-R, R) = (-\frac{1}{r}, \frac{1}{r})$

$\therefore$ radius of convergence $= \frac{1}{r} = R$.

Cor. Radius of convergence of $\sum a_n x^n$ is $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

Sol^n By Cauchy's th^m on limits, we have

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}, \text{ where } a_n > 0$$

$$\therefore R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad \left[\because \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}\right]$$

Q: Determine the radii of convergence of following power series (i) $\sum_{n=0}^{\infty} \frac{(\lfloor n)^2}{\lfloor 2n} x^{2n}$ (ii) $\frac{1}{2}x + \frac{1\cdot 3}{2\cdot 5}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}x^3 + \dots \infty$ (4)

Sol^n (i) Given series is $\sum_{n=0}^{\infty} \frac{(\lfloor n)^2}{\lfloor 2n} x^{2n}$

Taking $x^2 = t$, the series becomes $\sum_{n=0}^{\infty} \frac{(\lfloor n)^2}{\lfloor 2n} t^n = \sum a_n t^n$ (say)

where $a_n = \frac{(\lfloor n)^2}{\lfloor 2n}$ $\therefore a_{n+1} = \frac{(\lfloor n+1)^2}{\lfloor 2n+2}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{(\lfloor n)^2}{\lfloor 2n} \times \frac{\lfloor 2n+2}{(\lfloor n+1)^2} = \frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{2\left(2+\frac{1}{n}\right)}{1+\frac{1}{n}}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{2\left(2+\frac{1}{n}\right)}{1+\frac{1}{n}} = 4$$

$\therefore$ Given series is cgt for $|t| < 4$ i.e. $|x^2| < 4$

i.e. $|x| < 2$ i.e. $x \in (-2, 2)$

$\therefore$ radius of convergence $= 2$

(ii) Let given series is $\sum a_n x^n$, where $a_n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 5\cdot 8 \cdots (3n-1)}$

$\therefore a_{n+1} = \frac{1\cdot 3 \cdots (2n-1)(2n+1)}{2\cdot 5 \cdots (3n-1)(3n+2)}$ $\therefore \frac{a_{n+1}}{a_n} = \frac{2n+1}{3n+2} = \frac{2+\frac{1}{n}}{3+\frac{2}{n}}$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{3+\frac{2}{n}} = \frac{2}{3}.$$

$\therefore$ Radius of convergence $= \frac{3}{2}$

Q: Prove that series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ converges for $-1 < x \le 1$.

Sol^n The given power series is $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$

$\therefore a_n = \frac{(-1)^{n-1}}{2n-1}$ and $a_{n+1} = \frac{(-1)^n}{2n+1}$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n-1}{2n+1} = 1$$

$\therefore$ Radius of convergence $= 1$

∴ given series Converges for $-1 < x < 1$ (5)

For $\underline{x=1}$, series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$, which is alternating series of real no.

Since $\frac{1}{2n-1} > 0$ is monotonically decreasing and

$\lim_{n\to\infty} \frac{1}{2n-1} = 0$ ∴ by Leibnitz test $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ is cgt

Hence, given series Converges for $-1 < x \le 1$

Q: Find radius of Convergence and interval of Convergence of following power series.

(i) $\sum_{n=1}^{\infty} \lfloor n \, x^n$ (ii) $\sum \frac{z^{2n+1}}{2n+1}$ (iii) $\sum (3+4i)\, x^n$

(iv) $\sum \left(\frac{n\sqrt{2}+i}{1+2in}\right) x^n$ (v) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (vi) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

(vii) $\sum_{n=1}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} x^n$ (viii) $\sum \frac{(-1)^n}{n}(z-2i)^n$ (ix) $\sum_{n=1}^{\infty} \frac{\lfloor n}{n^n} x^n$.

(x) $\sum_{n=1}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} x^{2n}$ (xi) $\cancel{\sum_{n=1}^{\infty} \frac{\lfloor n}{n^n} x^n}$ $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(\lfloor n)^2 \, 2^{2n}}$

(xii) $\sum_{n=0}^{\infty} (x-1)^{n-1}$

Sol^n: (i) given power series is $\sum_{n=1}^{\infty} \lfloor n \, x^n = \sum_{n=1}^{\infty} a_n x^n$ (say)

∴ $a_n = \lfloor n$ and $a_{n+1} = \lfloor n+1$

∴ $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\lfloor n}{\lfloor n+1} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

∴ R = radius of convergence $= 0$.

and series has no interval of convergence

(ii) Given power series is $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{z}{2n+1}(z^2)^n$

Putting $z^2 = x$, given series becomes $\sum_{n=0}^{\infty} \frac{z}{2n+1} x^n = \sum_{n=0} a_n x^n$ (say)

∴ $a_n = \frac{z}{2n+1}$, $a_{n+1} = \frac{z}{2n+3}$

Then $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{2n+3}{2n+1} = 1$

$\therefore$ series $\sum_{n=0}^{\infty} \frac{3}{2n+1} x^n$ Converges for $|x|<1$

$\therefore$ given series $\sum_{n=0}^{\infty} \frac{3^{2n+1}}{2n+1}$ Converges for $|z^2|<1$

i.e. $|z|<1$ i.e. $z \in (-1,1)$

$\therefore$ $R =$ ~~Radius~~ Radius of Convergence $= 1$

and interval of convergence is $(-1,1)$

(iii) Given series is $\sum (3+4i) x^n = \sum a_n x^n$ (say)

$\therefore a_n = 3+4i \Rightarrow |a_n| = |3+4i| = \sqrt{9+16} = 5$

$$\therefore \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} 5^{1/n} = 5^0 = 1$$

$$\therefore R = \text{radius of convergence} = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}} = \frac{1}{1} = 1$$

and interval of convergence is $(-1,1)$

(iv) Given power series is $\sum \left(\frac{n\sqrt{2}+i}{1+2in}\right) x^n = \sum a_n x^n$

$$\therefore a_n = \frac{n\sqrt{2}+i}{1+2in} \Rightarrow |a_n| = \left|\frac{n\sqrt{2}+i}{1+2in}\right| = \frac{|n\sqrt{2}+i|}{|1+2in|} = \frac{\sqrt{2n^2+1}}{\sqrt{1+4n^2}}$$

$$\text{and } |a_{n+1}| = \frac{\sqrt{2(n+1)^2+1}}{\sqrt{1+4(n+1)^2}} = \frac{\sqrt{2n^2+4n+3}}{\sqrt{4n^2+8n+5}}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\sqrt{2n^2+1}}{\sqrt{1+4n^2}} \times \frac{\sqrt{4n^2+8n+5}}{\sqrt{2n^2+4n+3}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2+\frac{1}{n^2}}}{\sqrt{4+\frac{1}{n^2}}} \times \frac{\sqrt{4+\frac{8}{n}+\frac{5}{n^2}}}{\sqrt{2+\frac{4}{n}+\frac{3}{n^2}}} = \frac{\sqrt{2}}{2} \times \frac{2}{\sqrt{2}} = 1$$

$\therefore$ $R =$ radius of Convergence $= 1$

and interval of Convergence $= (-R,R) = (-1,1)$

(v) Given power series is $\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} a_n x^n$ (say)

$\therefore a_n = \frac{1}{n}$ and $a_{n+1} = \frac{1}{n+1}$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$$

$\therefore$ radius of convergence $(R) = 1$

and interval of convergence $(-R, R) = (-1, 1)$

(vi) Given power series is $\sum_{n=1}^{\infty} \frac{x^n}{n^n} = \sum_{n=1}^{\infty} a_n x^n$ (say)

$\therefore a_n = \frac{1}{n^n}$ $\therefore |a_n|^{1/n} = \left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \frac{1}{n}$.

$R =$ radius of convergence $= \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}} = \frac{1}{\lim_{n\to\infty} \frac{1}{n}} = \frac{1}{0} = \infty$.

and interval of convergence. $(-\infty, \infty)$

(vii) Given power series is $\sum_{n=1}^{\infty} \frac{\lfloor 2n}{(\lfloor n)^2} x^n = \sum_{n=1}^{\infty} a_n x^n$ (say)

$\therefore a_n = \frac{\lfloor 2n}{(\lfloor n)^2}$ and $a_{n+1} = \frac{\lfloor 2n+2}{(\lfloor n+1)^2}$

$$\Rightarrow \frac{a_n}{a_{n+1}} = \frac{(\lfloor n+1)^2}{\lfloor 2n+2} \times \frac{\lfloor 2n}{(\lfloor n)^2} = \frac{(n+1)^2 \cancel{(\lfloor n)^2}}{(2n+2)(2n+1)\cancel{\lfloor 2n}} \times \frac{\cancel{\lfloor 2n}}{\cancel{(\lfloor n)^2}}$$

$$= \frac{n+1}{2(2n+1)} = \frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)}$$

$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)} = \frac{1}{4}$

$\therefore$ radius of convergence $= \frac{1}{4} = 0.25$

or series converges for $|x| < R$ i.e. $|x| < .25$

i.e. $x \in (-0.25, 0.25)$ which is interval of convergence

(viii) Given power series is $\sum \frac{(-1)^n}{n}(z-2i)^n$

Put $z - 2i = x$

given series becomes $\sum \frac{(-1)^n}{n} x^n = \sum a_n x^n$ (say)

$\therefore a_n = \frac{(-1)^n}{n}$ and $a_{n+1} = \frac{(-1)^{n+1}}{n+1}$

$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$

$\therefore$ series $\sum \frac{(-1)^n}{n} x^n$ converges for $|x| < 1$

$\Rightarrow$ series $\sum \frac{(-1)^n}{n}(z-2i)^n$ converges for $|z-2i| < 1$

i.e. $-1 < z - 2i < 1$ i.e. $2i - 1 < z < 2i + 1$

$\therefore$ radius of convergence of given series $= 1$

and interval of convergence is $(2i-1, 2i+1)$

(ix) Given power series is $\sum_{n=1}^{\infty} \frac{\underline{|n}}{n^n} x^n = \sum_{n=1}^{\infty} a_n x^n$

$\therefore a_n = \frac{\underline{|n}}{n^n}$ and $a_{n+1} = \frac{\underline{|n+1}}{(n+1)^{n+1}}$

$\Rightarrow \left|\frac{a_n}{a_{n+1}}\right| = \frac{\underline{|n}}{\underline{|n+1}} \times \frac{(n+1)^{n+1}}{n^n} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Given series converges for $|x| < e$ i.e. $x \in (-e, e)$

$\therefore$ Radius of convergence $(R) = e$

and interval of convergence is $(-e, e)$

(x) Given power series is $\sum_{n=1}^{\infty} \frac{\underline{|2n}}{(\underline{|n})^2} x^{2n}$

Putting $x^2 = t$, power series becomes $\sum_{n=1}^{\infty} \frac{\underline{|2n}}{(\underline{|n})^2} t^n = \sum_{n=1}^{\infty} a_n t^n$ (say)

$\therefore a_n = \frac{\underline{|2n}}{(\underline{|n})^2}$ and $a_{n+1} = \frac{\underline{|2n+2}}{(\underline{|n+1})^2}$

$\Rightarrow \left|\frac{a_n}{a_{n+1}}\right| = \frac{(\underline{|n+1})^2}{\underline{|2n+2}} \times \frac{\underline{|2n}}{(\underline{|n})^2} = \frac{(n+1)^2}{(2n+2)(2n+1)}$

$= \frac{n+1}{2(2n+1)} = \frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)}$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{4}$

Series $\sum_{n=1}^{\infty} \frac{\underline{|2n}}{(\underline{|n})^2} t^n$ converges for $|t| < \frac{1}{4}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\underline{|2n}}{(\underline{|n})^2} x^{2n}$ converges for $|x^2| < \frac{1}{4}$

i.e. $|x| < \frac{1}{2}$ i.e. $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\therefore$ Radius of convergence $= \frac{1}{2}$

and interval of convergence $\left(-\frac{1}{2}, \frac{1}{2}\right)$

(xi) Given series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(\lfloor n)^2 2^{2n}} \cdot x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lfloor n)^2 2^{2n}} t^n$ (9)

where $x^2 = t$ $= \sum_{n=0}^{\infty} a_n t^n$ (say)

$\therefore a_n = \frac{(-1)^n}{(\lfloor n)^2 2^{2n}}$ and $a_{n+1} = \frac{(-1)^{n+1}}{(\lfloor n+1)^2 2^{2n+2}}$

$\Rightarrow \left|\frac{a_n}{a_{n+1}}\right| = \frac{(\lfloor n+1)^2 2^{2n+2}}{(\lfloor n)^2 2^{2n}} = 4(n+1)^2$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty$

power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(\lfloor n)^2 2^{2n}} t^n$ converges for $|t| < \infty$

$\Rightarrow$ given power series converges for $|x^2| < \infty$

i.e $|x| < \infty$ i.e. $x \in (-\infty, \infty)$.

$\therefore$ radius of convergence of given power series is ∞

and interval of convergence is $(-\infty, \infty)$

(xii) Given power series is $\sum_{n=0}^{\infty} (x-1)^{n-1} = \sum_{n=0}^{\infty} y^{n-1} = \sum_{n=1}^{\infty} y^n$

where $x-1 = y$. $= \sum_{n=1}^{\infty} a_n y^n$ (say)

$\therefore a_n = 1$ and $a_{n+1} = 1$

$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 1.$

$\therefore$ series $\sum_{n=1}^{\infty} y^n$ converges for $|y| < 1$ i.e. $-1 < y < 1$

$\Rightarrow$ series $\sum_{n=0}^{\infty} (x-1)^{n-1}$ converges for $|x-1| < 1$

i.e. $-1 < x-1 < 1$ i.e. $0 < x < 2$

$\therefore$ Radius of convergence of given power series is

1 and interval of convergence is $(0, 2)$

Q: Find interval of convergence of following power series (10)

(i) $\sum n^2 x^n$ (ii) $\sum \left(\frac{2^n}{n^2}\right) x^n$ (iii) $\sum \left(\frac{3^n}{4^n}\right) \frac{x^n}{n}$ (iv) $\sum \frac{n 4^n}{3^n} x^n$.

Solⁿ (i) Given power series is $\sum n^2 x^n = \sum a_n x^n$ (say)

$\therefore \; a_n = n^2 \Rightarrow \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n^2)^{1/n}$

$= \lim_{n\to\infty} (n^{\frac{1}{n}})^2 = 1^2 = 1$

$\Rightarrow R = \dfrac{1}{\lim_{n\to\infty} |a_n|^{\frac{1}{n}}} = \frac{1}{1} = 1$

Given series converges for $|x| < R$ i.e. $|x| < 1$ i.e.

$x \in (-1, 1)$

(ii) Given series is $\sum \frac{2^n}{n^2} x^n = \sum a_n x^n$ (say)

$\therefore \; a_n = \frac{2^n}{n^2}$ and $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{2}{(n^{\frac{1}{n}})^2} = 2$

$\Rightarrow R = \dfrac{1}{\lim_{n\to\infty} |a_n|^{\frac{1}{n}}} = \frac{1}{2}$

$\therefore$ given power series converges for $|x| < R$ i.e. $|x| < \frac{1}{2}$

i.e. $x \in (-\frac{1}{2}, \frac{1}{2})$

(iii) Given series is $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \frac{x^n}{n} = \sum_{n=1}^{\infty} a_n x^n$ (say)

$\therefore \; a_n = \left(\frac{3}{4}\right)^n \times \frac{1}{n}$ and $a_{n+1} = \left(\frac{3}{4}\right)^{n+1} \times \frac{1}{n+1}$

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3}{4} \times \frac{n}{n+1} = \frac{3}{4}$.

$\Rightarrow R = \dfrac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \frac{4}{3}$

Given power series converges for $|x| < R$ i.e. $|x| < \frac{4}{3}$

$\Rightarrow$ i.e $x \in (-\frac{4}{3}, \frac{4}{3})$

(iv) same as (iii) part

Q: Find the values for which power series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n}$ converges

Sol^n: Let $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n} = \sum_{n=1}^{\infty} \frac{y^n}{n} = \sum_{n=1}^{\infty} a_n y^n$ (say)

where $x+2=y$. and $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{y^n}{n}$ Converges for $|y|<1$ i.e. $-1<y<1$

$\Rightarrow$ given series converges for $|x+2|<1$ i.e.

$-1<x+2<1$ i.e. $-3<x<-1$

Q: Find radius of convergence of $\sum_{n=2}^{\infty} \frac{(x-2)^{n-2}}{n \log n}$.

Sol^n: Let $\sum_{n=2}^{\infty} \frac{(x-2)^{n-2}}{n\log n} = \sum_{n=2}^{\infty} \frac{y^{n-2}}{n\log n}$ where $y=x-2$

$= \sum_{m=0}^{\infty} \frac{1}{(m+2)\log(m+2)} \cdot y^m$ $\left(\begin{array}{l}\text{Put } n-2=m \\ \text{i.e. } n=m+2\end{array}\right)$

$= \sum_{m=0}^{\infty} a_m y^m$ (say.)

$\therefore a_m = \frac{1}{(m+2)\log(m+2)}$ and $a_{m+1} = \frac{1}{(m+3)\log(m+3)}$

$\therefore \lim_{n\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \lim_{n\to\infty} \frac{m+2}{m+3} \times \lim_{m\to\infty} \frac{\log(m+2)}{\log(m+3)} = 1 \times \lim_{m\to\infty} \frac{\frac{1}{m+2}}{\frac{1}{m+3}}$

$= \lim_{m\to\infty} \frac{m+3}{m+2} = 1$

If R be radius of convergence of power series

$\frac{1}{R} = \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = 1 \Rightarrow R=1$

$\therefore$ Radius of convergence $= 1$

Q. Find radius of convergence of power series

$$1 + \frac{a \cdot b}{1 \cdot c} x + \frac{a(a+1)\, b(b+1)}{1 \cdot 2\, c(c+1)} x^2 + \cdots$$

Sol^n: Given power series is

$$\sum_{n=0}^{\infty} \frac{[a(a+1)\cdots(a+\overline{n-1})][b(b+1)\cdots(b+\overline{n-1})]}{[1\cdot2\cdot3\cdots n][c(c+1)\cdots(c+\overline{n-1})]} x^n = \sum_{n=0}^{\infty} u_n x^n \text{ (say)}$$

$$\therefore u_n = \frac{[a(a+1)\cdots(a+\overline{n-1})][b(b+1)\cdots(b+\overline{n-1})]}{[1\cdot2\cdots n][c(c+1)\cdots(c+\overline{n-1})]}$$

and $$u_{n+1} = \frac{[a(a+1)\cdots(a+\overline{n-1})(a+n)][b(b+1)\cdots(b+\overline{n-1})(b+n)]}{[1\cdot2\cdot3\cdots n(n+1)][c(c+1)\cdots(c+\overline{n-1})(c+n)]}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(n+1)(c+n)}{(a+n)(b+n)} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)} = 1$$

$\therefore$ radius of convergence $(R) = 1$ and $(-1, 1)$ is interval of convergence

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÷ Uniform Convergence of Power Series ÷

Th^m: Prove that power series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly cgt for $|x| \le \rho < R$, where R is radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Proof:– Let ρ' be any real no. s.t. $\rho < \rho' < R$

Then series $\sum_{n=0}^{\infty} a_n x^n$ is cgt for $|x| = \rho'$ [$\because \sum a_n x^n$ is cgt for $|x| < R$ and $\rho' < R$]

i.e. $\sum_{n=0}^{\infty} a_n (\rho')^n$ is cgt hence bdded

$\therefore$ $\exists$ the real no K s.t

$$|a_n (\rho')^n| \le K \quad \forall n \in N \quad \text{——— (1)}$$

$\therefore$ for $|x| \le \rho$, we have

$$|a_n x^n| = \left|a_n(\rho')^n \left(\frac{x}{\rho'}\right)^n\right| = |a_n(\rho')^n| \left|\left(\frac{x}{\rho'}\right)^n\right|$$

$$\le K\left(\frac{\rho}{\rho'}\right)^n = M_n \text{ say.} \quad \left[\begin{array}{l}\text{by (1) and} \\ |x| \le \rho\end{array}\right]$$

Now series $\sum M_n = K \sum_{n=0}^{\infty} \left(\frac{\rho}{\rho'}\right)^n$ is cgt, being a G.P. series with CR $\frac{\rho}{\rho'} < 1$ $\quad$ [$\because \rho < \rho' \Rightarrow \frac{\rho}{\rho'} < 1$]

Hence by W.M test for uniform convergence of series of function.

Power series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly cgt for $|x| \le \rho < R$.

i.e. with in interval of convergence

---o---

Th^m: Prove that series obtained by differentiating a power series term by term has the same radius of convergence as the original series.

Proof: The power series $\sum_{n=0}^{\infty} a_n x^n$ after differentiating term by term becomes $\sum_{n=1}^{\infty} n a_n x^{n-1}$. ~~tet~~

Let R' be radius of convergence of new series.

$$\therefore \frac{1}{R'} = \lim_{n\to\infty} |n a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(n^{1/n} \cdot |a_n|^{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R} \qquad \left[\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right]$$

$$\Rightarrow R' = R$$

Thus the differentiated series has same radius of convergence as the original power series.

Hence a power series can be differentiated term by term at any point $x \in (-R, R)$.

Thm: A power series $\sum_{n=0}^{\infty} a_n x^n$ can be integrated term by term so long as the limits of integration lie strictly with in range $(-R, R)$, where R is radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Proof: The series $\sum_{n=0}^{\infty} a_n x^n$ after integrating term by term becomes $\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$.

let R' be radius of convergence of new series.

Then, $$\frac{1}{R'} = \lim_{n\to\infty} \left|\frac{a_n}{n+1}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|a_n|^{\frac{1}{n}}}{(n+1)^{1/n}}$$

$$= \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R} \qquad \left[\because \lim_{n\to\infty} (n+1)^{\frac{1}{n}} = 1\right]$$

$$\Rightarrow R' = R$$

Let $l = \lim_{n\to\infty} (n+1)^{\frac{1}{n}}$

$\Rightarrow \log l = \lim_{n\to\infty} \frac{\log(n+1)}{n} \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{1} = 0$

$\Rightarrow \log l = 0 \Rightarrow l = 1$

Thus radius of convergence of series obtained by integrating term by term has same radius of convergence as original series.

$\Rightarrow \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ is uniformly cgt on $(-R, R)$ [by thm pg (13)]

Hence power series can be integrated term by term with in the range $(-R, R)$

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Q: Write properties of a power series.

Sol^n Let R be radius of convergence of power series $\sum_{n=0}^{\infty} a_n x^n$ then following properties holds

(i) A power series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly cgt with in $(-R, R)$ i.e. with in interval of convergence.

(ii) A power series $\sum_{n=0}^{\infty} a_n x^n$ is continuous function of x with in $(-R, R)$.

(iii) A power series can be integrated term by term so long as limits of integration lies with in range $(-R, R)$.

(iv) A power series can be differentiated term by term ~~so long~~ with in $(-R, R)$.

<u>Abel's Summability</u>: A series of real no $\sum_{n=0}^{\infty} a_n$ is said to be Abel summable to a value S, if the associated power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $0 \leq x < 1$ to a function f and $\lim_{x \to 1^-} f(x) = S$.

eg. Consider series $\sum_{n=0}^{\infty} a_n$, where $a_n = (-1)^n$

Here associated power series is

$$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots \quad \text{---} (1)$$

which converges to $f(x) = \frac{1}{1+x}$ for $0 \leq x < 1$. $\left[\text{Using } S = \frac{a}{1-r} \right]$

Now $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$

Hence $\sum_{n=0}^{\infty} (-1)^n$ is Abel summable to $\frac{1}{2}$

Abel's Inequality: If a seq $\{v_n\}$ of +ve terms is monotonic decreasing and h, H are lower and upper limits of partial sums $a_1, a_1+a_2, a_1+a_2+a_3, \ldots,$ ~~then~~ $a_1+a_2+\cdots+a_p$, then

$$h v_1 < \sum_{n=1}^{p} < H v_1$$

Abel's thm : **Statement**: If series of real no $\sum_{n=0}^{\infty} a_n$ is cgt and has sum S, then associated power series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly cgt for $0 \le x \le 1$ and

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = S.$$

Proof: Since series of real no $\sum_{n=0}^{\infty} a_n$ is cgt

$\therefore$ by Cauchy's general principal for convergence of series of real no.

given $\epsilon > 0$ $\exists$ $m \in \mathbb{N}$ s.t

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon \quad \forall\ n \ge m,\ p \ge 1 \quad \text{—(1)}$$

Now seq $\{x^n\}$ is +ve term. and monotonic decreasing seq for $0 \le x \le 1$ $\therefore$ by Abel's inequality

$$|a_n x^n + a_{n+1} x^{n+1} + \cdots + a_{n+p} x^{n+p}| \le \epsilon x^n \le \epsilon \quad \text{—(2)}$$

By Cauchy's cretrion for uniform convergence [$\because$ $0 \le x \le 1$] of series of funtion

$\sum_{n=0}^{\infty} a_n x^n$ is unifomly cgt for $0 \le x \le 1$.

$\therefore$ $\sum_{n=0}^{\infty} a_n x^n$ is Continuous function of x in $0 \le x \le 1$

$$\therefore \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \lim_{x \to 1^-} a_n x^n = \sum_{n=0}^{\infty} a_n \times 1 = S$$

$\left[\because \sum_{n=0}^{\infty} a_n = S \text{ given}\right]$

Tauber's thm: If series $\sum_{n=0}^{\infty} a_n$ is Abel summable to s and $\lim_{n\to\infty} n a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ converges to s.

or

If $n a_n \to 0$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n \to s$ as $x \to 1^-$, then $\sum_{n=0}^{\infty} a_n$ converges to s.

Proof: Since $\lim_{n\to\infty} n a_n = 0$ —— ①

$\therefore$ By Cauchy's first thm on limits, we have

$$\lim_{n\to\infty} \frac{|a_1| + 2.|a_2| + 3|a_3| + \cdots + n|a_n|}{n} = 0 \text{ —— ②}$$

Also, $\sum_{n=0}^{\infty} a_n$ is Abel summable to s

$\therefore \lim_{x\to 1^-} f(x) = s$, where $\sum_{n=0}^{\infty} a_n x^n = f(x)$ for $0 \le x < 1$

i.e. $\lim_{n\to\infty} f\left(1-\frac{1}{n}\right) = s$ —— ③

from ①, ② and ③, for given $\epsilon > 0$, there exist $m \in N$ s.t for $n \geq m$, we have

$$|n a_n| < \frac{\epsilon}{3} \quad \text{i.e. } |a_n| < \frac{\epsilon}{3n} \text{ —— ④}$$

$$\left|\frac{|a_1| + 2|a_2| + 3|a_3| + \cdots + n|a_n|}{n}\right| < \frac{\epsilon}{3} \text{ —— ⑤}$$

and $$\left| f\left(1-\frac{1}{n}\right) - s\right| < \frac{\epsilon}{3} \text{ —— ⑥}$$

Let $S_n = a_1 + a_2 + \cdots + a_n$ be nth partial sum of $\sum a_n$

Then
$$\begin{aligned}
|S_n - s| &= |S_n - s + f(x) - f(x)| \\
&= \left| f(x) - s + a_0 + \sum_{r=1}^{n} a_r - a_0 - \sum_{r=1}^{\infty} a_r x^r \right| \\
&= \left| f(x) - s + \sum_{r=1}^{n} a_r - \sum_{r=1}^{\infty} a_r . x^r \right| \\
&= \left| f(x) - s + \sum_{r=1}^{n} a_r (1 - x^r) - \sum_{r=n+1}^{\infty} a_r x^r \right| \\
&\le |f(x) - s| + \left| \sum_{r=1}^{n} a_r (1 - x^r) \right| + \left| \sum_{r=n+1}^{\infty} a_r x^r \right|
\end{aligned}$$

$$\Rightarrow |S_n - s| \le |f(x) - s| + (1-x)\left|\sum_{r=1}^{n} r a_r\right| + \sum_{r=n+1}^{\infty} |a_r| x^r \quad \text{--- (7)}$$

Now using (4), for $n \ge m$, we have.

$$\sum_{r=n+1}^{\infty} |a_r| x^r \le \sum_{r=n+1}^{\infty} \frac{\epsilon}{3n} x^r = \frac{\epsilon}{3n} \cdot \frac{x^{n+1}}{1-x} < \frac{\epsilon}{3n(1-x)}$$

$\left[\because 0 \le x < 1 \text{ and } |a_r| < \frac{\epsilon}{3r} < \frac{\epsilon}{3n} \text{ as } r > n\right]$

$\therefore$ from (7), for $n \ge m$, we have

$$|S_n - s| \le |f(x) - s| + (1-x)\sum_{r=1}^{n} r|a_r| + \frac{\epsilon}{3n(1-x)}$$

Putting $x = 1 - \frac{1}{n}$, we get

$$|S_n - s| \le \left|f\left(1-\tfrac{1}{n}\right) - s\right| + \left(\tfrac{1}{n}\right)\sum_{r=1}^{n} r|a_r| + \frac{\epsilon}{3}$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall\, n \ge m \quad [\text{by (5) and (6)}]$$

$\Rightarrow \quad S_n \to s \text{ as } n \to \infty$

Hence $\sum_{n=0}^{\infty} a_n$ converges to s.

÷ Some Important Expansions ÷

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \cdots\cdots$
2. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
3. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$
5. $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \frac{x^7}{7} + \cdots$
6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$
7. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$
8. $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$

Q: From the eqⁿ $\tan^{-1}x = \int_0^x \frac{1}{1+x^2}\,dx$

Obtain Gregory's series $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{---}$

within what range of x does this hold?

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{---}$

Solⁿ: We have $(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \text{---} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ ——— ①

$= \sum_{n=0}^{\infty} (-1)^n t^n$ [Taking $x^2 = t$]

Here $a_n = (-1)^n$, $a_{n+1} = (-1)^{n+1}$

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Power series $\sum_{n=0}^{\infty} (-1)^n t^n$ converges for $|t| < 1$

$\Rightarrow \sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges for $|x^2| < 1 \Rightarrow |x| < 1$

i.e. $-1 < x < 1$

$\Rightarrow$ Power series on RHS of ① is uniformly cgt for $|x| < 1$

$\therefore$ Integrate both side of ① term by term from 0 to x, we get

$$\int_0^x \frac{dx}{1+x^2} = \int_0^x 1\cdot dx - \int_0^x x^2\,dx + \int_0^x x^4\,dx \text{ ----}$$

or $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \text{----}$ for $-1 < x < 1$ ——— ②

For $x = \pm 1$ power series on RHS of ② becomes

$\pm\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{—}\right) = \pm (-1)^{n-1}\frac{1}{2n-1}$ which is alternating series

$\therefore$ by Leibnitz's test, power series on RHS of ② is cgt for $x = \pm 1$ also.

$\therefore$ it converges in $[-1,1]$ and hence converges uniformly for $x \in [-1,1]$.

Thus $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $-1 \le x \le 1$

$\therefore \tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Q. Show that $\int_0^1 \frac{\tan^{-1}x}{x}\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)^2}$.

Sol^n: We have

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } |x| \le 1$$

(Prove it)

$$\Rightarrow \frac{\tan^{-1}x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots$$

Integrating both side wrt x from 0 to 1, we have

$$\int_0^1 \frac{\tan^{-1}x}{x}\,dx = \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots\right]_0^1$$

$$= 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)^2}$$

Q: Show that $\int_0^x \tan^{-1}x\,dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n(2n-1)}$ for $|x|<1$.

Show that the result also holds for $x=1$ and deduce that

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = 0.4388$$

Sol^n: We have $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $|x| \le 1$

(Prove it)

Integrating both side w.r.t 'x' from 0 to x, we get

$$\int_0^x \tan^{-1}x\,dx = \left[\frac{x^2}{2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \cdots\right]_0^x$$

$$\Rightarrow \int_0^x \tan^{-1}x\,dx = \frac{x^2}{1\cdot 2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \cdots \quad (1)$$

$$\therefore \int_0^x \tan^{-1}x\,dx = \sum_{n=1}^{\infty} (-1)^{n-1}\cdot \frac{x^{2n}}{2n(2n-1)}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{2n(2n-1)} \quad [\text{Taking } x^2 = t]$$

$$= \sum_{n=1}^{\infty} u_n\, t^n.$$

Here $u_n = (-1)^{n-1} \dfrac{1}{2n(2n-1)}$ and $u_{n+1} = \dfrac{(-1)^n}{(2n+2)(2n+1)}$

$$\therefore \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{(2-\frac{1}{n})2}{(2+\frac{1}{n})(2+\frac{2}{n})} = 1$$

$\therefore$ Power series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{2n(2n-1)} t^n$ converges for $|t| < 1$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^{2n}}{2n(2n-1)}$ Converges for $|x^2| < 1$ i.e $|x| < 1$
i.e. $-1 < x < 1$

At $x = 1$, the power series on RHS of (1) becomes an alternating series which is cgt by Leibnitz test. Thus (1) holds for $x = 1$ also.

Now $\displaystyle\int_0^x \tan^{-1}x\,dx = \int_0^x \underset{\text{II}}{1}\cdot \underset{\text{I}}{\tan^{-1}x}\,dx$ (Integrating by parts)

$$= \left[(\tan^{-1}x).x\right]_0^x - \int_0^x \frac{x}{1+x^2}dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2) \quad (2)$$

from (1) and (2), we have.

$$\frac{x^2}{1\cdot 2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \cdots = x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

Put $x=1$, we have $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \cdots = \tan^{-1}1 - \frac{1}{2}\log_e 2$

$$\Rightarrow \left(1-\frac{1}{2}\right) - \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) = \frac{\pi}{4} - \frac{1}{2}\,\frac{(.3010)}{.4343} \quad [\because \pi = 3.1416]$$

$$= 0.7854 - \frac{1}{2}(0.6931) = 0.4388$$

$$\Rightarrow 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = 0.4388$$

Q: Show that $\int_0^{} \frac{\sin^{-1}x}{x}\,dx = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$ for $|x| \le 1$.

Sol^n: We have

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}(x^2) + \frac{1\cdot 3}{2\cdot 4}(x^2)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(x^2)^3 + \cdots$$

$$\cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}(x^2)^n + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} x^{2n}$$

$$\therefore \frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} t^n \quad \text{—— (1)}$$

where $x^2 = t$

Here $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$, $a_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots(2n)(2n+2)}$

$$\therefore R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2n+2}{2n+1} = 1$$

$\therefore$ Power series on RHS of (1) is cgt for $|t|<1$
i.e. for $|x^2|<1$ i.e. for $|x|<1$

For $x^2=1$, the series on RHS of (1) becomes.

$$1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} = 1 + \sum u_n \quad \text{(say)}$$

Here $u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$, $u_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)}$

$\therefore \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$ $\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1}-1\right)$

$= \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$

$\therefore$ By Raabe's test power series on RHS for $x^2=1$ is dgt

$\because$ Series on RHS of ① is uniformly cgt in $-1<x<1$.

The integrated series will have same characteristics.

Thus integrating ① both side of ① wrt x, we get

$$\sin^{-1}x = x + \sum_{n=0}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} + c$$

for $x=0$, $c=0$

$$\therefore \sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7} + \cdots$$

or $\sin^{-1}x = x + \sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$ for $-1<x<1$ ——— ②

$= \sum v_n$ (say)

for $x=1$ $\frac{v_n}{v_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}$

$$\therefore \lim_{n\to\infty} n\left(\frac{v_n}{v_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{4n^2+10n+6-4n^2-4n-1}{4n^2+4n+1}\right] = \lim_{n\to\infty}\frac{n(6n+5)}{4n^2+4n+1}$$

$$= \frac{6}{4} > 1.$$

$\therefore$ Series on RHS of ② is cgt for $x=1$ by Raabe's test

IIly " " ② " for $x=-1$

$\therefore$ series on RHS of (2) is uniformly cgt for $-1 \le x \le 1$

$$\therefore \sin^{-1}x = x + \sum_{n=1}^{\infty} \frac{1.3.5---(2n-1)}{2.4.6---(2n)} \frac{x^{2n+1}}{2n+1} \text{ for } -1 \le x \le 1$$

Dividing both side by x, we get

$$\frac{\sin^{-1}x}{x} = 1 + \frac{1}{2}\cdot\frac{x^2}{3} + \frac{1.3}{2.4}\cdot\frac{x^4}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^6}{7} + ---$$
$$--- + \frac{1.3.5---(2n-1)}{2.4.6---2n}\cdot\frac{x^{2n}}{2n+1} + --- \text{ for } |x| \le 1$$

Integrating both side w.r.t x from 0 to 1, we have

$$\int_0^1 \frac{\sin^{-1}x}{x}\,dx = \Big[x + \frac{1}{2}\cdot\frac{x^3}{(3)^2} + \frac{1.3}{2.4}\cdot\frac{x^5}{(5)^2} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{(7)^2} + --$$
$$+ \frac{1.3.5--(2n-1)}{2.4.6--2n}\cdot\frac{x^{2n+1}}{(2n+1)^2} + --\Big]_0^1$$

$$= 1 + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2} + --- + \frac{1.3.5--(2n-1)}{2.4.6---2n}\cdot\frac{1}{(2n+1)^2}$$
$$+ --$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1.3.5--(2n-1)}{2.4.6--2n}\cdot\frac{1}{(2n+1)^2} \text{ for } |x| \le 1$$

Q: Find the sum for $|x| \le 1$ of the power series

$$x + \sum_{n=1}^{\infty} (-1)^n \frac{1.3.5---(2n-1)}{2.4.6--(2n)}\cdot\frac{x^{2n+1}}{2n+1}$$ and deduce that

$$1 + \frac{1.3}{2.4}\cdot\frac{1}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{9} + --- = \frac{\pi}{4} + \frac{1}{2}(1+\sqrt{2})$$

Soln :- We have

$$(1+x^2)^{-1/2} = 1 - \frac{1}{2}(x^2) + \frac{1.3}{2.4}(x^2)^2 - \frac{1.3.5}{2.4.6}(x^2)^3 + -$$
$$---+ \frac{1.3.5---(2n-1)}{2.4.6---(2n)}(-1)^n(x^2)^n + -- \quad (1)$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n \frac{1.3.5---(2n-1)}{2.4.6--(2n)}(x^2)^n = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{1.3.--(2n-1)}{2.4--(2n)} t^n$$
(where $x^2 = t$)

Here $a_n = (-1)^n \dfrac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$, $a_{n+1} = (-1)^{n+1}\dfrac{1\cdot3\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)}$

$\therefore \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\dfrac{2n+2}{2n+1} = 1$

$\therefore$ Power series on RHS of ① is cgt for $|t|<1$

i.e $|x^2|<1$ i.e. $|x|<1$

further for $x^2=1$, power series on RHS becomes an alternating series and is cgt by Leibnitz Test

$\therefore$ Power series on RHS is uniformly cgt in $[-1,1]$

Now integrating ① both side wrt 'x', we have

$$\log|x+\sqrt{1+x^2}| = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7} + \cdots + (-1)^n\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} + \cdots + c$$

for $x=0$, $c=0$.

$$\therefore x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5} - \cdots + (-1)^n\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} + \cdots = \log|x+\sqrt{1+x^2}| \quad \text{for } |x|\le1$$

i.e. $$x + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} = \log|x+\sqrt{1+x^2}| \quad \text{for } |x|\le1$$

When $x=1$

$$1 - \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7} + \cdots = \log(1+\sqrt{2}) \quad \text{—— ②}$$

Further $$1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7} + \cdots = \frac{\pi}{2} \quad \text{—— ③}$$

[$\because \sin^{-1}x = x + \sum_{n=1}^{\infty}\dfrac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\cdot\dfrac{x^{2n+1}}{2n+1}$ for $-1\le x\le1$.

Putting $x=1$, we obtained eqⁿ ③]

Adding ② and ③, we get

$$2\left[1+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{1}{9}+\cdots\right]=\frac{\pi}{2}+\log(1+\sqrt{2})$$

$$\Rightarrow 1+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{1}{9}+\cdots=\frac{\pi}{4}+\frac{1}{2}\log(1+\sqrt{2})$$

Q: Show that $\int_0^x \log(1+x)\,dx=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+1}}{n(n+1)}$ for $|x|<1$. Does result hold for $x=\pm1$?

Sol^n: We know that

$$(1+x)^{-1}=1-x+x^2-x^3+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n \quad \text{—— ①}$$

here $a_n=(-1)^n$, $a_{n+1}=(-1)^{n+1}$

$$\therefore\ R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=1$$

$\therefore$ Power series on RHS of ① converges uniformly in $(-1,1)$

At $x=\pm1$, power series on RHS of ① becomes

$\sum_{n=0}^{\infty}(-1)^n$ and $\sum_{n=0}^{\infty}(-1)^{n+1}$ which oscillates between 0 and 1

Now integrating ① both side w.r.t 'x', we have.

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+C$$

for $x=0$, $C=0$

$$\therefore\ \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \quad \text{for } -1<x<1 \quad \text{—— ②}$$

for $x=1$, Power series on RHS of ② becomes

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum(-1)^{n-1}\cdot\frac{1}{n}$, which is an alternating series and cgt by Leibnitz test.

For $x=-1$, Power series on RHS of ② becomes

$-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\right)=-\sum\frac{1}{n}$ which is dgt by p-test

$$\therefore\ \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \text{ for } -1<x\le1 \quad \text{—— ③}$$

Integrating (3) both side w.r.t 'x' from 0 to x,

$$\int_0^x \log(1+x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \frac{x^5}{4\cdot 5} + \cdots\right]_0^x$$

$$\Rightarrow \int_0^x \log(1+x)\,dx = \frac{x^2}{1\cdot 2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \frac{x^5}{4\cdot 5} + \cdots \quad \text{--- (4)}$$

$$\Rightarrow \int_0^x \log(1+x)\,dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+1}}{n(n+1)} \quad \text{for} \quad |x|<1$$

For $x=1$, series on RHS of (4) becomes $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n(n+1)}$ which is alternating series and is cgt by Leibnitz test.

$\therefore$ Result holds for $x=1$.

For $x=-1$, series on RHS of (4) becomes

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots = \sum \frac{1}{n(n+1)} = \sum a_n$$

$\therefore$ $a_n = \frac{1}{n(n+1)}$ let $b_n = \frac{1}{n^2}$

$\therefore$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

By comparison test $\sum a_n$ and $\sum b_n$ behave alike

But $\sum b_n = \sum \frac{1}{n^2}$ is cgt by p-test

$\Rightarrow$ $\sum a_n$ i.e. series on RHS of (4) at $x=-1$ is cgt

$\therefore$ Result holds for $x=-1$

Q: Show that

(i) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ for $-1 \le x < 1$

(ii) $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

(iii) $\frac{1}{2}\left[\log(1-x)\right]^2 = \frac{x^2}{2} + \frac{x^3}{3}\left(1+\frac{1}{2}\right) + \frac{x^4}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \cdots$
for $-1 \le x < 1$

Sol^n: We know that

$(1-x)^{-1} = 1+x+x^2+x^3+ \cdots$ ——— ①

which is G.P. series with C.R x, hence cgt for $|x|<1$

i.e. for $-1<x<1 \Rightarrow$ series is uniformly cgt for $|x|<1$

also Power series on RHS of ① does not converge for $x=\pm 1$

Integrating ① wrt x, we get

$$\frac{\log(1-x)}{-1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + c$$

$$\Rightarrow \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad \left[\begin{matrix}\text{For } x=0\\ c=0\end{matrix}\right]$$

for $-1<x<1$ ——— ②

At $x=-1$, power series on RHS of ②

reduces to $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \sum (-1)^{n-1}\frac{1}{n}$

which is an alternating series and hence cgt

by Leibnitz test.

At $x=1$, series on RHS of ② $= -\sum\frac{1}{n}$ which

is dgt by p-test.

$$\therefore \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \text{ for } -1\le x<1$$

——— ③

(iii) From ① and ③, both power series on RHS.

converges in $(-1,1)$ $[-1<x<1]$

$\therefore$ Their Cauchy product is cgt in $(-1,1)$ i.e. $-1<x<1$

$$\therefore (1-x)^{-1}\log(1-x) = \left[1+x+x^2+x^3+\cdots\right]\left[-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots\right]$$

$$= -x - x^2\left(1+\frac{1}{2}\right) - x^3\left(1+\frac{1}{2}+\frac{1}{3}\right) - \cdots \text{ for } -1<x<1$$

Integrating both side wrt x from 0 to x, we get

$$-\frac{1}{2}\left[\log(1-x)\right]^2 = -\frac{x^2}{2} - \frac{x^3}{3}\left(1+\frac{1}{2}\right) - \cdots \quad \text{for } -1<x<1$$

$$\Rightarrow \frac{1}{2}\left[\log(1-x)\right]^2 = \frac{x^2}{2} + \frac{x^3}{3}\left(1+\frac{1}{2}\right) + \frac{x^4}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \cdots$$

for $-1<x<1$

At $x=-1$, power series on R.H.S becomes

$$\frac{1}{2} - \frac{1}{3}\left(1+\frac{1}{2}\right) + \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \cdots$$

Which is alternating series and is cgt by Leibnitz test.

The power series on R.H.S does not converge for $x=1$

$$\therefore \frac{1}{2}\left[\log(1-x)\right]^2 = \frac{x^2}{2} + \frac{x^3}{3}\left(1+\frac{1}{2}\right) + \frac{x^4}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \cdots$$

for $-1 \le x < 1$.

(ii) Put $x=-1$ in ③, we get

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Q: Prove that

$$\int_0^1 \frac{1-t}{1-xt^3}\,dt = \frac{1}{1\cdot 2} + \frac{x}{4\cdot 5} + \frac{x^2}{7\cdot 8} + \frac{x^3}{10\cdot 11} + \cdots \quad \text{for } -1 \le x \le 1$$

Deduce that

(i) $\dfrac{\pi}{3\sqrt{3}} = \dfrac{1}{1\cdot 2} + \dfrac{1}{4\cdot 5} + \dfrac{1}{7\cdot 8} + \cdots$

(ii) $\dfrac{\pi}{6\sqrt{3}} + \dfrac{1}{3}\log 2 = \dfrac{1}{1\cdot 2} + \dfrac{1}{7\cdot 8} + \dfrac{1}{13\cdot 14} + \cdots$

Soln: We have

$$(1-t)(1-xt^3)^{-1} = (1-t)\left[1 + xt^3 + x^2t^6 + x^3t^9 + \cdots\right]$$

$$= (1-t) + x(t^3-t^4) + x^2(t^6-t^7) + x^3(t^9-t^{10}) + \cdots$$

$$\Rightarrow \int_0^1 \frac{1-t}{1-xt^3}\,dt = \left[\left(t-\frac{t^2}{2}\right) + x\left(\frac{t^4}{4}-\frac{t^5}{5}\right) + x^2\left(\frac{t^7}{7}-\frac{t^8}{8}\right) + \cdots\right]_0^1$$

$$= \left[\left(1-\frac{1}{2}\right) + x\left(\frac{1}{4}-\frac{1}{5}\right) + x^2\left(\frac{1}{7}-\frac{1}{8}\right) + \cdots\right]$$

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$$= \frac{1}{1\cdot 2} + \frac{x}{4\cdot 5} + \frac{x^2}{7\cdot 8} + \cdots + \frac{x^{n-1}}{(3n-2)(3n-1)} + \cdots \quad \text{---}(1)$$

for power series on RHS of (1)

$$a_n = \frac{1}{(3n-2)(3n-1)}, \quad a_{n+1} = \frac{1}{(3n+1)(3n+2)}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(3n-2)(3n-1)}{(3n+1)(3n+2)} = \lim_{n\to\infty} \frac{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)}{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)}$$

$$= 1$$

$\therefore$ Power series of RHS of (1) Converges for $|x|<1$

i.e. $-1<x<1$

for $x=1$, we have

$$n\left[\frac{a_n}{a_{n+1}} - 1\right] = n\left[\frac{(3n+1)(3n+2)}{(3n-2)(3n-1)} - 1\right] = \frac{18n^2}{9n^2-9n+2}$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = 2 > 1$$

$\therefore$ Power series on RHS of (1) Converges for $x=1$ by Raabe's Test.

Again for $x=-1$, power series on RHS becomes an alternating series and is cgt by Leibnitz test.

Thus

$$\int_0^1 \frac{1-t}{1-xt^3}\,dt = \frac{1}{1\cdot 2} + \frac{x}{4\cdot 5} + \frac{x^2}{7\cdot 8} + \cdots \quad \text{for } -1 \le x \le 1 \quad \text{---}(2)$$

(i) Put $x=1$ in (2), we get

$$\frac{1}{1\cdot 2} + \frac{1}{4\cdot 5} + \frac{1}{7\cdot 8} + \cdots \int_0^1 \frac{1-t}{1-t^3}\,dt = \int_0^1 \frac{dt}{t^2+t+1}$$

$$= \int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\sqrt{3}/2}\left[\tan^{-1}\frac{\left(t+\frac{1}{2}\right)}{\sqrt{3}/2}\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1 = \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{2}{\sqrt{3}} \times \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}.$$

$$\therefore \frac{\pi}{3\sqrt{3}} = \frac{1}{1\cdot 2} + \frac{1}{4\cdot 5} + \frac{1}{7\cdot 8} + \cdots \qquad \text{—— (3)}$$

(ii) Put $x = -1$ in (2), we get,

$$\frac{1}{1\cdot 2} - \frac{1}{4\cdot 5} + \frac{1}{7\cdot 8} - \frac{1}{10\cdot 11} + \cdots = \int_0^1 \frac{1-t}{1+t^3}\,dt \qquad \text{—— (4)}$$

Now $\dfrac{1-t}{1+t^3} = \dfrac{1-t}{(1+t)(t^2-t+1)} = \dfrac{A}{1+t} + \dfrac{Bt+C}{t^2-t+1}$

$$\Rightarrow 1-t = A(t^2-t+1) + (Bt+C)(1+t) \qquad \text{—— (5)}$$

Put $1+t=0$ i.e. $t=-1$ in (5), we have.

$$2 = A(1+1+1) \Rightarrow \boxed{A = \frac{2}{3}}$$

Coff of t^2) $\quad 0 = A+B \Rightarrow \boxed{B = -A = -\frac{2}{3}}$

Coff of t) $\quad -1 = -A+B+C \Rightarrow -1 = -\frac{2}{3} - \frac{2}{3} + C$

$$\Rightarrow \boxed{C = \frac{1}{3}}$$

$$\therefore \frac{1-t}{1+t^3} = \frac{\frac{2}{3}}{1+t} + \frac{-\frac{2}{3}t + \frac{1}{3}}{t^2-t+1}$$

$$\Rightarrow \int_0^1 \frac{1-t}{1+t^3}\,dt = \int_0^1 \left[\frac{2}{3} \times \frac{1}{1+t} - \frac{1}{3}\,\frac{2t-1}{t^2-t+1}\right]dt$$

$$= \left[\frac{2}{3}\log(1+t) - \frac{1}{3}\log(t^2-t+1)\right]_0^1 = \frac{2}{3}\log 2$$

From eqn (4), we have

$$\frac{1}{1\cdot 2} - \frac{1}{4\cdot 5} + \frac{1}{7\cdot 8} - \frac{1}{10\cdot 11} + \frac{1}{13\cdot 14} - \cdots = \frac{2}{3}\log 2 \qquad \text{—— (6)}$$

(3) + (6), we have

$$2\left[\frac{1}{1\cdot 2}+\frac{1}{7\cdot 8}+\frac{1}{13\cdot 14}+\cdots\right]=\frac{\pi}{3\sqrt{3}}+\frac{2}{3}\log 2$$

$$\Rightarrow \frac{1}{1\cdot 2}+\frac{1}{7\cdot 8}+\frac{1}{13\cdot 14}+\cdots=\frac{\pi}{6\sqrt{3}}+\frac{1}{3}\log 2.$$

Q: Show that

$$1-\frac{1}{5}-\frac{1}{7}+\frac{1}{11}+\frac{1}{13}-\frac{1}{17}-\frac{1}{19}+\cdots=\int_0^1\frac{1-x^4}{1+x^6}\,dx=\frac{1}{\sqrt{3}}\log(2+\sqrt{3})$$

Soln:

$$\frac{1-x^4}{1+x^6}=(1-x^4)(1+x^6)^{-1}=(1-x^4)(1-x^6+x^{12}-x^{18}+\cdots)$$

$$=1-x^4-x^6+x^{10}+x^{12}-x^{16}-x^{18}+x^{22}+\cdots$$

Integrating both side wrt x from '0' to '1', we have

$$\int_0^1\frac{1-x^4}{1+x^6}\,dx=\left[x-\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^{11}}{11}+\frac{x^{13}}{13}-\frac{x^{17}}{17}-\frac{x^{19}}{19}+\cdots\right]_0^1$$

$$\Rightarrow \int_0^1\frac{1-x^4}{1+x^6}\,dx=1-\frac{1}{5}-\frac{1}{7}+\frac{1}{11}+\frac{1}{13}-\frac{1}{17}-\frac{1}{19}+\cdots \quad (1)$$

Again $$\int\frac{1-x^4}{1+x^6}\,dx=\int\frac{(1+x^2)(1-x^2)}{(1+x^2)(x^4-x^2+1)}\,dx=\int\frac{1-x^2}{x^4-x^2+1}\,dx$$

$$=-\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}\,dx=-\int\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-3}\,dx$$

Put $x+\frac{1}{x}=t$

$\left(1-\frac{1}{x^2}\right)dx=dt$

$$=-\int\frac{1}{t^2-3}\,dt=\int\frac{dt}{(\sqrt{3})^2-t^2}=\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right|$$

$$=\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+x+\frac{1}{x}}{\sqrt{3}-x-\frac{1}{x}}\right|=\frac{1}{2\sqrt{3}}\log\left|\frac{x^2+\sqrt{3}x+1}{-x^2+\sqrt{3}x-1}\right|$$

$$\therefore \int_0^1\frac{1-x^4}{1+x^6}\,dx=\frac{1}{2\sqrt{3}}\log\left|\frac{x^2+\sqrt{3}x+1}{-x^2+\sqrt{3}x-1}\right|_0^1$$

$$=\frac{1}{2\sqrt{3}}\log\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

$$\frac{2+\sqrt{3}}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\frac{(2+\sqrt{3})^2}{4-3}=(2+\sqrt{3})^2$$

$$=\frac{1}{2\sqrt{3}}\log(2+\sqrt{3})^2=\frac{1}{\sqrt{3}}\log(2+\sqrt{3}) \quad (2)$$

from (1) and (2)

$$1-\frac{1}{5}-\frac{1}{7}+\frac{1}{11}+\frac{1}{13}-\frac{1}{17}-\frac{1}{19}+\cdots=\int_0^1\frac{1-x^4}{1+x^6}\,dx=\frac{1}{\sqrt{3}}\log(2+\sqrt{3})$$